# *Bayesian Compressive Sensing Framework for High-Dimensional Surrogate Construction*

*K. Sargsyan*[1], C. Safta[1], D. Ricciuto[2],
B.Debusschere[1],H. Najm[1],P. Thornton[2]

[1]Sandia National Laboratories
Livermore, CA

[2]Oak Ridge National Laboratory
Oak Ridge, TN

*Sponsored by DOE, Biological and Environmental Research,*
*under Accelerated Climate Modeling for Energy (ACME).*

*Sandia National Laboratories is a multi-program laboratory operated by Sandia Corporation,*
*a wholly owned subsidiary of Lockheed Martin Corporation,*
*for the U.S. Department of Energy's National Nuclear Security Administration*
*under contract DE-AC04-94AL85000.*

# OUTLINE

- Surrogates needed for complex models
- Polynomial Chaos (PC) surrogates do well with uncertain inputs
- Bayesian regression provide results with uncertainty certificate
- Compressive sensing ideas deal with high-dimensionality

# Surrogate construction: scope and challenges

Construct surrogate for a complex model $f(\boldsymbol{\lambda})$ to enable

- Global sensitivity analysis
- Optimization
- Forward uncertainty propagation
- Input parameter calibration
- $\cdots$

- Computationally expensive model simulations, data sparsity
  - Need to build accurate surrogates with as few training runs as possible
- High-dimensional input space
  - Too many samples needed to cover the space
  - Too many terms in the polynomial expansion

# Surrogate construction: scope and challenges

Construct surrogate for a complex model $f(\boldsymbol{\lambda})$ to enable

- Global sensitivity analysis
- Optimization
- Forward uncertainty propagation
- Input parameter calibration
- $\cdots$

- Computationally expensive model simulations, data sparsity
  - Need to build accurate surrogates with as few training runs as possible
- High-dimensional input space
  - Too many samples needed to cover the space
  - Too many terms in the polynomial expansion

# Surrogate construction: scope and challenges

Construct surrogate for a complex model $f(\boldsymbol{\lambda})$ to enable

- Global sensitivity analysis
- Optimization
- Forward uncertainty propagation
- Input parameter calibration
- $\cdots$

- Computationally expensive model simulations, data sparsity
  - Need to build accurate surrogates with as few training runs as possible
- High-dimensional input space
  - Too many samples needed to cover the space
  - Too many terms in the polynomial expansion

# Polynomial Chaos surrogate

- Build/presume PC for input parameter $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}(\boldsymbol{x}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{x})$$

# Polynomial Chaos surrogate

- Build/presume PC for input parameter $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}(\boldsymbol{x}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{x})$$

- E.g., gaussian with known moments $\mu_i, \sigma_i$,

$$\lambda_i = \mu_i + \sigma_i x_i$$

# Polynomial Chaos surrogate

- Build/presume PC for input parameter $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}(\boldsymbol{x}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{x})$$

- Input parameters are represented via their cumulative distribution function $F(\cdot)$, such that, with $x_i \sim$ Uniform$[-1, 1]$

$$\lambda_i = F_{\lambda_i}^{-1}\left(\frac{x_i + 1}{2}\right), \qquad\qquad \text{for } i = 1, 2, \ldots, d.$$

# Polynomial Chaos surrogate

- Build/presume PC for input parameter $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}(\boldsymbol{x}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{x})$$

- If input parameters are uniform in $[a_i, b_i]$, then

$$\lambda_i = \frac{a_i + b_i}{2} + \frac{b_i - a_i}{2}\, x_i$$

# Polynomial Chaos surrogate

- Build/presume PC for input parameter $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}(\boldsymbol{x}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{x})$$

- If input parameters are uniform in $[a_i, b_i]$, then

$$\lambda_i = \frac{a_i + b_i}{2} + \frac{b_i - a_i}{2}\, x_i$$

- Forward function $f(\cdot)$, output $u$

$$u = f(\boldsymbol{\lambda}(\boldsymbol{x})) \qquad\qquad u = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x}) \equiv g(\boldsymbol{x})$$

- Global sensitivity information for free
  - Sobol indices, variance-based decomposition.

# Alternative methods to obtain PC coefficients

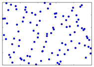
$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Projection $\qquad c_k = \frac{\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle}{\langle \Psi_k^2(\boldsymbol{x})\rangle}$

  The integral $\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle = \int u(\boldsymbol{x})\Psi_k(\boldsymbol{x})d\boldsymbol{x}$ can be estimated by

  - Monte-Carlo

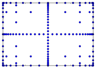
$$\frac{1}{N}\sum_{j=1}^{N} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)$$



many(!) random samples

  - Quadrature

$$\sum_{j=1}^{Q} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)w_j$$



samples at quadrature

# Alternative methods to obtain PC coefficients

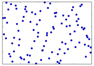
$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Projection $\qquad c_k = \frac{\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle}{\langle \Psi_k^2(\boldsymbol{x})\rangle}$

  The integral $\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle = \int u(\boldsymbol{x})\Psi_k(\boldsymbol{x})d\boldsymbol{x}$ can be estimated by

  - Monte-Carlo

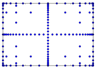
$$\frac{1}{N}\sum_{j=1}^{N} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)$$



many(!) random samples

  - Quadrature

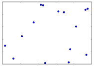
$$\sum_{j=1}^{Q} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)w_j$$



samples at quadrature

- Bayesian regression

$$P(c_k|u(\boldsymbol{x}_j)) \propto P(u(\boldsymbol{x}_j)|c_k)P(c_k)$$



any (number of) samples

# Alternative methods to obtain PC coefficients

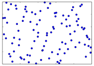
$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Projection $\qquad c_k = \frac{\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle}{\langle \Psi_k^2(\boldsymbol{x})\rangle}$

  The integral $\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle = \int u(\boldsymbol{x})\Psi_k(\boldsymbol{x})d\boldsymbol{x}$ can be estimated by

  - Monte-Carlo

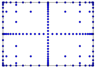
$$\frac{1}{N}\sum_{j=1}^{N} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)$$



many(!) random samples

  - Quadrature

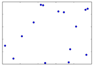
$$\sum_{j=1}^{Q} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)w_j$$



samples at quadrature

- Bayesian regression

$$\underbrace{P(\boldsymbol{c}|\mathcal{D})}_{\text{Posterior}} \propto \underbrace{P(\mathcal{D}|\boldsymbol{c})}_{\text{Likelihood}} \underbrace{P(\boldsymbol{c})}_{\text{Prior}}$$



any (number of) samples

# Bayesian inference of PC surrogate

$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x}) \equiv g_{\boldsymbol{c}}(\boldsymbol{x}) \qquad \overbrace{P(\boldsymbol{c}|\mathcal{D})}^{\text{Posterior}} \propto \overbrace{P(\mathcal{D}|\boldsymbol{c})}^{\text{Likelihood}} \overbrace{P(\boldsymbol{c})}^{\text{Prior}}$$

- <u>Data</u> consists of *training runs*

$$\mathcal{D} \equiv \{(\boldsymbol{x}_i, u_i)\}_{i=1}^{N}$$

- <u>Likelihood</u> with a gaussian noise model with $\sigma^2$ fixed or inferred,

$$L(\boldsymbol{c}) = P(\mathcal{D}|\boldsymbol{c}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^{N} \exp\left(-\frac{(u_i - g_{\boldsymbol{c}}(\boldsymbol{x}))^2}{2\sigma^2}\right)$$

- <u>Prior</u> on $\boldsymbol{c}$ is chosen to be conjugate, uniform or gaussian.
- <u>Posterior</u> is a *multivariate normal*

$$\boldsymbol{c} \quad \in \quad \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

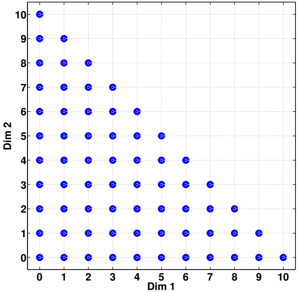
- The (uncertain) surrogate is *a gaussian process*

$$\sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x}) = \boldsymbol{\Psi}(\boldsymbol{x})^T \boldsymbol{c} \quad \in \quad \mathcal{GP}(\boldsymbol{\Psi}(\boldsymbol{x})^T \boldsymbol{\mu}, \boldsymbol{\Psi}(\boldsymbol{x}) \boldsymbol{\Sigma} \boldsymbol{\Psi}(\boldsymbol{x}')^T)$$

# Bayesian inference of PC surrogate: high-d, low-data regime

$$y = u(\boldsymbol{x}) \approx \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

$$\Psi_k(x_1, x_2, ..., x_d) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_d}(x_d)$$

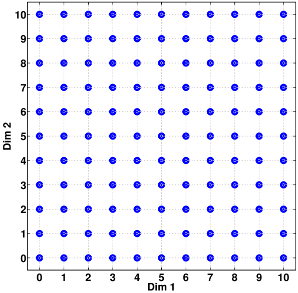



- Issues:
  - how to properly choose the basis set?
  - need to work in underdetermined regime $N < K$: fewer data than bases (d.o.f.)
- Discover the underlying low-d structure in the model
  - get help from the machine learning community

# Bayesian inference of PC surrogate: high-d, low-data regime

$$y = u(\boldsymbol{x}) \approx \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

$$\Psi_k(x_1, x_2, ..., x_d) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_d}(x_d)$$

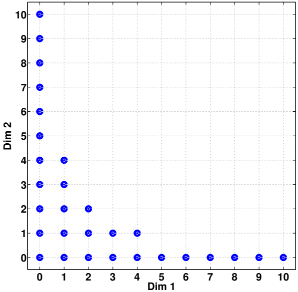



- Issues:
  - how to properly choose the basis set?
  - need to work in underdetermined regime $N < K$: fewer data than bases (d.o.f.)
- Discover the underlying low-d structure in the model
  - get help from the machine learning community

# Bayesian inference of PC surrogate: high-d, low-data regime

$$y = u(\boldsymbol{x}) \approx \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

$$\Psi_k(x_1, x_2, ..., x_d) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_d}(x_d)$$

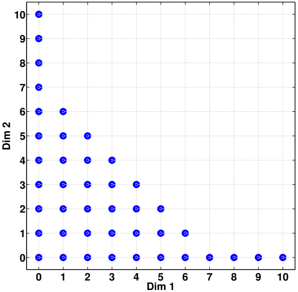



- Issues:
  - how to properly choose the basis set?
  - need to work in underdetermined regime $N < K$: fewer data than bases (d.o.f.)
- Discover the underlying low-d structure in the model
  - get help from the machine learning community

# Bayesian inference of PC surrogate: high-d, low-data regime

$$y = u(\boldsymbol{x}) \approx \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

$$\Psi_k(x_1, x_2, ..., x_d) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_d}(x_d)$$

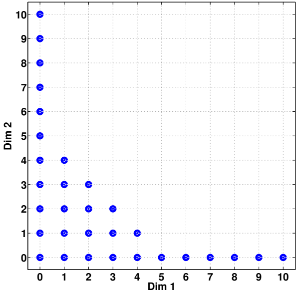



- Issues:
    - how to properly choose the basis set?
    - need to work in underdetermined regime $N < K$: fewer data than bases (d.o.f.)
- Discover the underlying low-d structure in the model
    - get help from the machine learning community

# Bayesian inference of PC surrogate: high-d, low-data regime

$$y = u(\boldsymbol{x}) \approx \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

$$\Psi_k(x_1, x_2, ..., x_d) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_d}(x_d)$$




- Issues:
  - how to properly choose the basis set?
  - need to work in underdetermined regime $N < K$: fewer data than bases (d.o.f.)
- Discover the underlying low-d structure in the model
  - get help from the machine learning community

# In a different language....

- $N$ training data points $(\boldsymbol{x}_n, u_n)$ and $K$ basis terms $\Psi_k(\cdot)$
- Projection matrix $\boldsymbol{P}^{N\times K}$ with $\boldsymbol{P}_{nk} = \Psi_k(\boldsymbol{x}_n)$
- Find regression weights $\boldsymbol{c} = (c_0, \ldots, c_{K-1})$ so that

$$\boldsymbol{u} \approx \boldsymbol{P}\boldsymbol{c} \qquad \text{or} \qquad u_n \approx \sum_k c_k \Psi_k(\boldsymbol{x}_n)$$

- The number of polynomial basis terms grows fast; a $p$-th order, $d$-dimensional basis has a total of $K = (p+d)!/(p!d!)$ terms.
- For limited data and large basis set ($N < K$) this is a sparse signal recovery problem $\Rightarrow$ need some regularization/constraints.

- Least-squares $\quad$ *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2\}$
- The 'sparsest' $\quad$ *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_0\}$
- Compressive sensing $\quad$ *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_1\}$

# In a different language....

- $N$ training data points $(\boldsymbol{x}_n, u_n)$ and $K$ basis terms $\Psi_k(\cdot)$
- Projection matrix $\boldsymbol{P}^{N\times K}$ with $\boldsymbol{P}_{nk} = \Psi_k(\boldsymbol{x}_n)$
- Find regression weights $\boldsymbol{c} = (c_0, \ldots, c_{K-1})$ so that

$$\boldsymbol{u} \approx \boldsymbol{P}\boldsymbol{c} \qquad \text{or} \qquad u_n \approx \sum_k c_k \Psi_k(\boldsymbol{x}_n)$$

- The number of polynomial basis terms grows fast; a $p$-th order, $d$-dimensional basis has a total of $K = (p+d)!/(p!d!)$ terms.
- For limited data and large basis set ($N < K$) this is a sparse signal recovery problem $\Rightarrow$ need some regularization/constraints.

- Least-squares *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2\}$
- The 'sparsest' *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_0\}$
- Compressive sensing *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_1\}$

Bayesian Likelihood Prior

# Bayesian Compressive Sensing (BCS), or Relevance Vector Machine (RVM)

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\alpha) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2)p(\sigma_k^2|\alpha)d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter $\alpha$ can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{P}^T\boldsymbol{u} \qquad \boldsymbol{\Sigma} = \sigma^2(\boldsymbol{P}^T\boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

[Tipping, 2001, Ji *et al.*, 2008; Babacan *et al.*, 2010]

# Bayesian Compressive Sensing (BCS), or Relevance Vector Machine (RVM)

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad\qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\alpha) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2)p(\sigma_k^2|\alpha)d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter $\alpha$ can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{P}^T\boldsymbol{u} \qquad\qquad \boldsymbol{\Sigma} = \sigma^2(\boldsymbol{P}^T\boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

[Tipping, 2001, Ji *et al.*, 2008; Babacan *et al.*, 2010]

# Bayesian Compressive Sensing (BCS), or Relevance Vector Machine (RVM)

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\alpha) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2) p(\sigma_k^2|\alpha) d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter $\alpha$ can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2} \boldsymbol{\Sigma} \boldsymbol{P}^T \boldsymbol{u} \qquad \boldsymbol{\Sigma} = \sigma^2 (\boldsymbol{P}^T \boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

[Tipping, 2001, Ji *et al.*, 2008; Babacan *et al.*, 2010]

# Bayesian Compressive Sensing (BCS), or Relevance Vector Machine (RVM)

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad\qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\alpha) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2) p(\sigma_k^2|\alpha) d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter $\alpha$ can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2} \boldsymbol{\Sigma} \boldsymbol{P}^T \boldsymbol{u} \qquad\qquad \boldsymbol{\Sigma} = \sigma^2 (\boldsymbol{P}^T \boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

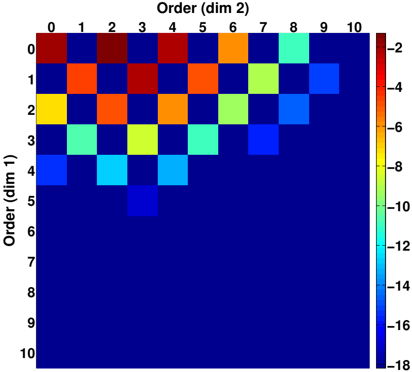
- KEY: Some $\sigma_k^2 \to 0$, hence the corresponding basis terms are dropped.

[Tipping, 2001, Ji *et al.*, 2008; Babacan *et al.*, 2010]

# BCS removes unnecessary basis terms

$$f(x,y) = \cos(x + 4y)$$

$$f(x,y) = \cos(x^2 + 4y)$$



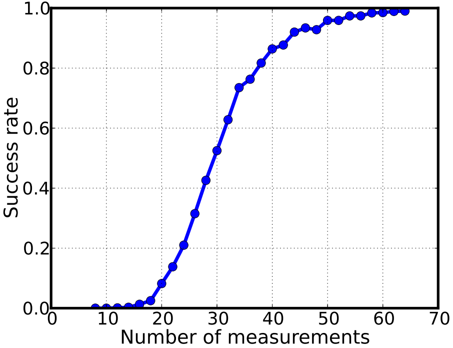

# Success rate grows with more data and 'sparser' model

Consider test function

$$f(\boldsymbol{x}) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

where only $S$ coefficients $c_k$ are non-zero. Typical setting is

$$S < N < K$$

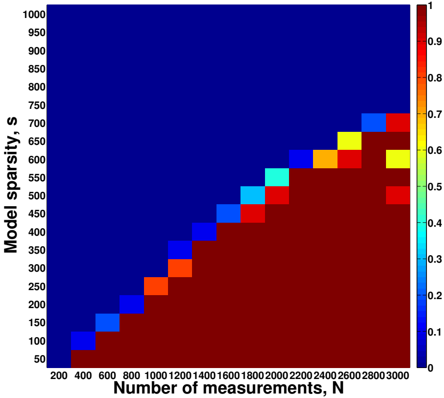


Consider test function

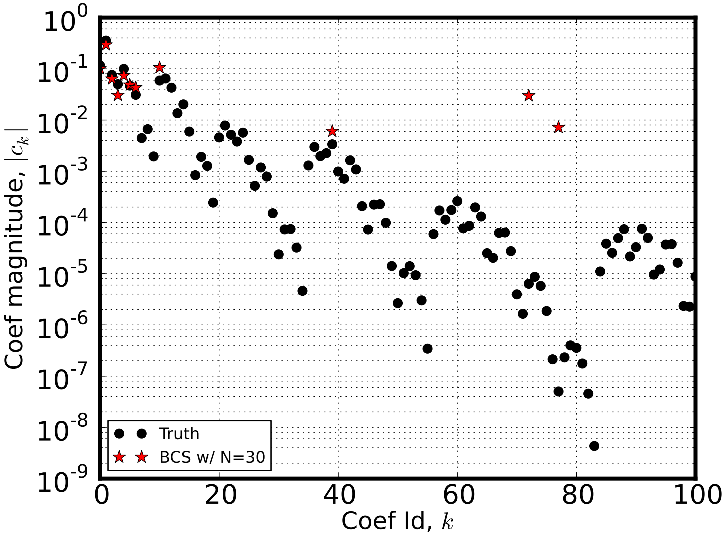
$$f(\boldsymbol{x}) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

where only $S$ coefficients $c_k$ are non-zero. Typical setting is

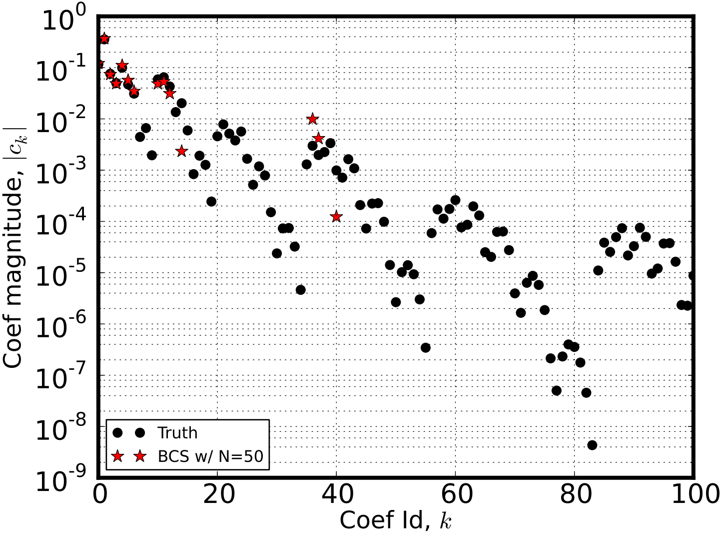
$$S < N < K$$



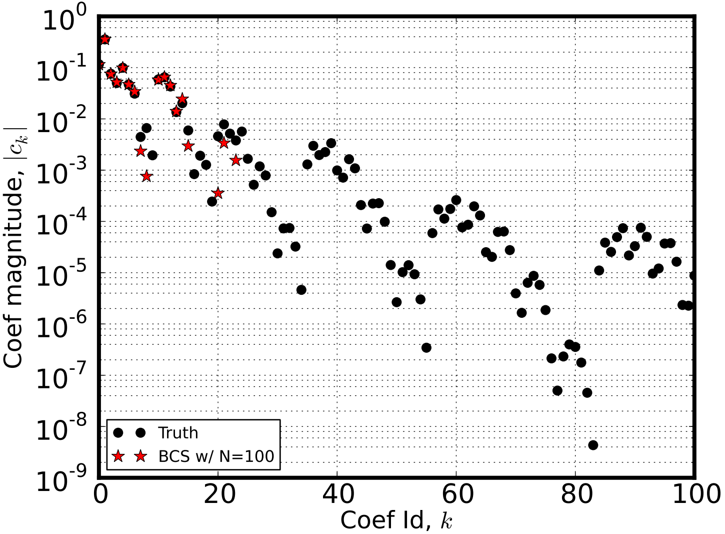
# BCS recovers true PC coefficients with increased number of measurements



# BCS recovers true PC coefficients with increased number of measurements



# BCS recovers true PC coefficients with increased number of measurements



# Bayesian Compressive Sensing

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad\qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\boldsymbol{\alpha}) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2) p(\sigma_k^2|\alpha) d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter $\alpha$ can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{P}^T\boldsymbol{u} \qquad\qquad \boldsymbol{\Sigma} = \sigma^2(\boldsymbol{P}^T\boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

- KEY: Some $\sigma_k^2 \to 0$, hence the corresponding basis terms are dropped.

# Weighted Bayesian Compressive Sensing

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad\qquad p(\sigma_k^2|\alpha_k) = \frac{\alpha_k}{2} e^{-\frac{\alpha_k \sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\boldsymbol{\alpha}) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2) p(\sigma_k^2|\alpha_k) d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha_k}}{2} e^{-\sqrt{\alpha_k}|c_k|}$$

- The parameter $\alpha_k$ can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha_k, \sigma^2$ and allows exact Bayesian solution

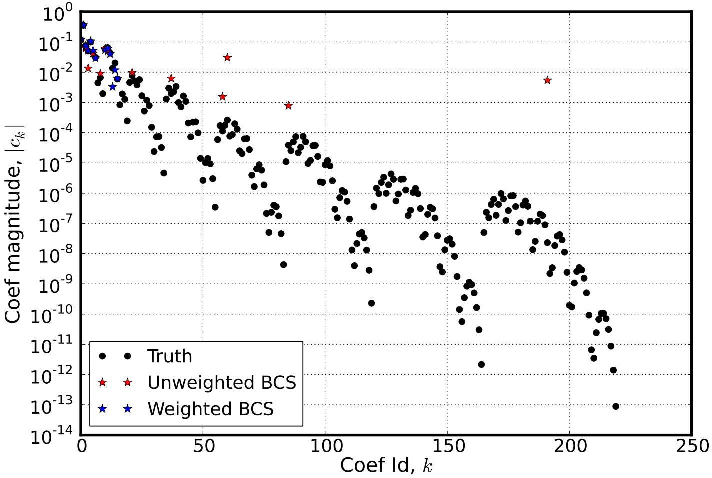
$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2} \boldsymbol{\Sigma} \boldsymbol{P}^T \boldsymbol{u} \qquad\qquad \boldsymbol{\Sigma} = \sigma^2 (\boldsymbol{P}^T \boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

- KEY: Some $\sigma_k^2 \to 0$, hence the corresponding basis terms are dropped.

# WBCS recovers true coefficients better




$$f(\boldsymbol{x}) = x_0 \cos\left(e + \sum_{i=1}^{9} x_i/i\right)$$

# Iteratively reweighting Compressive Sensing [Candes *et al.*, 2007]

Sparsest solution: $min||\boldsymbol{c}||_0$ such that $\boldsymbol{u} \approx \boldsymbol{P}\boldsymbol{c}$

Compressive sensing: $min||\boldsymbol{c}||_1$ such that $\boldsymbol{u} \approx \boldsymbol{P}\boldsymbol{c}$

Weighted compressive sensing: $min||\boldsymbol{W}\boldsymbol{c}||_1$ such that $\boldsymbol{u} \approx \boldsymbol{P}\boldsymbol{c}$

# Iteratively reweighting Compressive Sensing [Candes *et al.*, 2007]

Sparsest solution: $min||\boldsymbol{c}||_0$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$

Compressive sensing: $min||\boldsymbol{c}||_1$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$

Weighted compressive sensing: $min||\boldsymbol{Wc}||_1$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$

For sparse signals, $\boldsymbol{u} = \boldsymbol{Pc}^s$, with $||\boldsymbol{c}_s||_0 = S < K$, ideal weights are

$$\boldsymbol{W} = diag\left(\frac{1}{|c_k^s|}\right)$$

[i.e., $W_{kk} = +\infty$ if $c_k^s = 0$]

In practice, the true signal coefficients are not known, so...

# Iteratively reweighting Compressive Sensing [Candes *et al.*, 2007]

| | |
|---|---|
| Sparsest solution: | $min\|\|\boldsymbol{c}\|\|_0$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$ |
| Compressive sensing: | $min\|\|\boldsymbol{c}\|\|_1$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$ |
| Weighted compressive sensing: | $min\|\|\boldsymbol{Wc}\|\|_1$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$ |

For sparse signals, $\boldsymbol{u} = \boldsymbol{Pc}^s$, with $||\boldsymbol{c}_s||_0 = S < K$, ideal weights are

$$\boldsymbol{W} = diag\left(\frac{1}{|c_k^s|}\right)$$ [i.e., $W_{kk} = +\infty$ if $c_k^s = 0$]

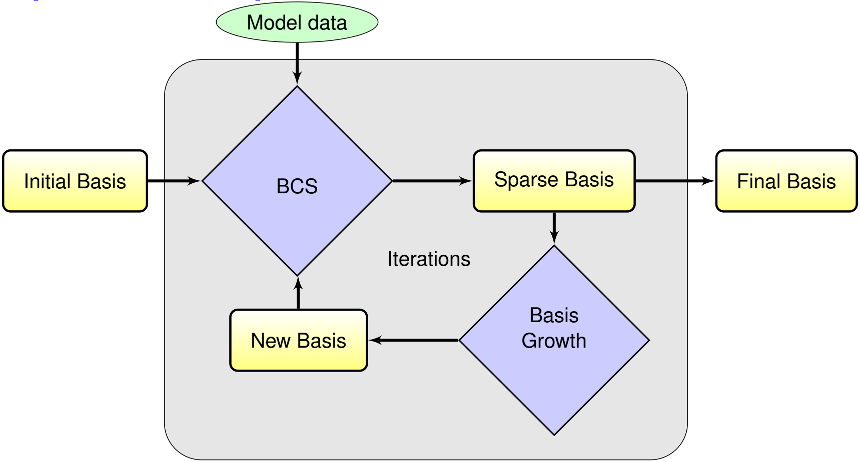
In practice, the true signal coefficients are not known, so...

Iterative re-weighting

$$\boldsymbol{W}^{(i+1)} = diag\left(\frac{1}{|c_k^{(i)}| + \epsilon}\right)$$ [$\epsilon \ll 1$ for stability]

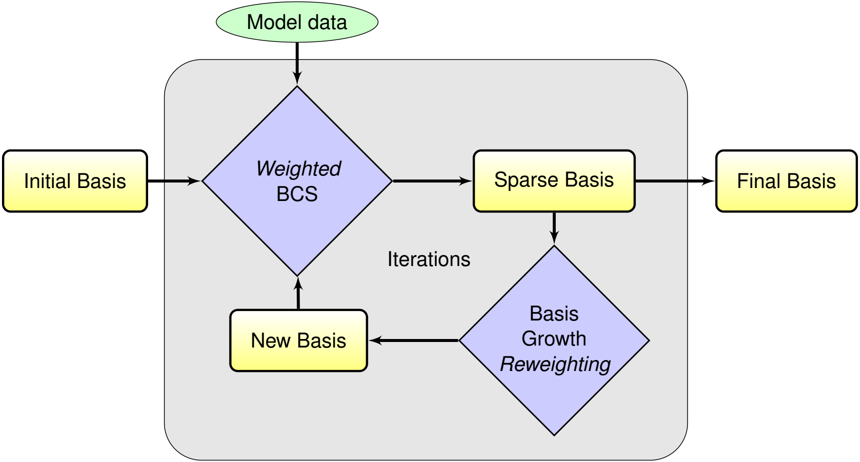
# Iterative Bayesian Compressive Sensing (iBCS)

- *Iterative BCS*: We implement an iterative procedure that allows increasing the order for the relevant basis terms while maintaining the dimensionality reduction [Sargsyan *et al.* 2014]. In a pure CS setting, [Jakeman *et al.* 2015].

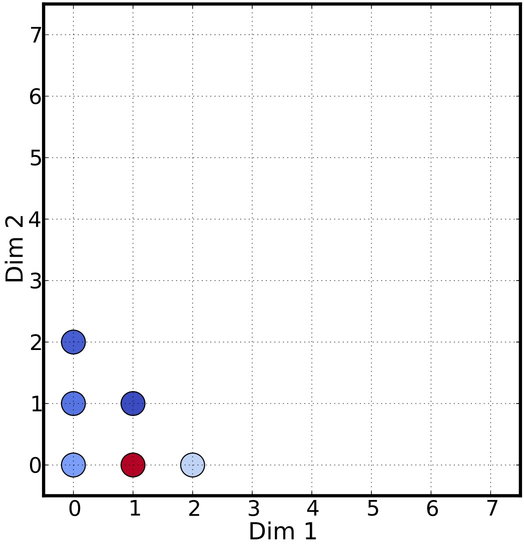


# Iterative Bayesian Compressive Sensing (iBCS)

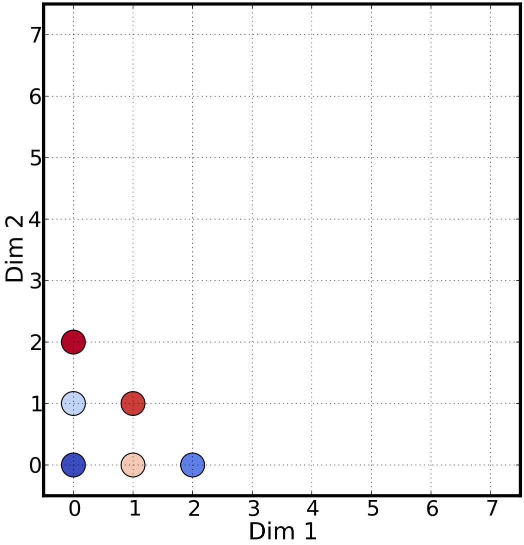
- Combine basis growth and reweighting!



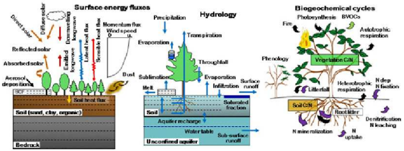
# Basis set growth: simple anisotropic function

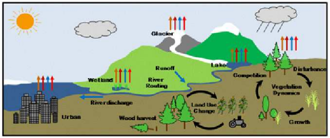


# Basis set growth: ... added outlier term



# Application of Interest: Community Land Model



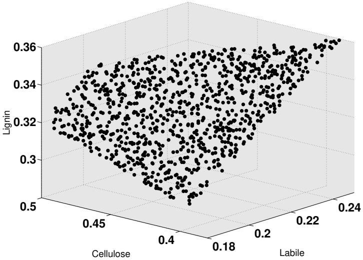

**http://www.cesm.ucar.edu/models/clm/**

- Nested computational grid hierarchy
- A single-site, $1000$-yr simulation takes $\sim 10$ hrs on $1$ CPU
- Involves $\sim 50$ input parameters; some dependent
- Non-smooth input-output relationship

# Input correlations: Rosenblatt transformation

- Rosenblatt transformation maps any (not necessarily independent) set of random variables $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ to uniform i.i.d.'s $\{x_i\}_{i=1}^d$ [Rosenblatt, 1952].

$$
\begin{aligned}
x_1 &= F_1(\lambda_1) \\
x_2 &= F_{2|1}(\lambda_2|\lambda_1) \\
x_3 &= F_{3|2,1}(\lambda_3|\lambda_2, \lambda_1) \\
&\vdots \\
x_d &= F_{d|d-1,\ldots,1}(\lambda_d|\lambda_{d-1}, \ldots, \lambda_1)
\end{aligned}
$$




- Inverse Rosenblatt transformation $\boldsymbol{\lambda} = R^{-1}(\boldsymbol{x})$ ensures a well-defined input PC construction

$$
\lambda_i = \sum_{k=0}^{K-1} \lambda_{ik} \Psi_k(\boldsymbol{x})
$$

- Caveat: the conditional distributions are often hard to evaluate accurately.

# Piecewise PC expansion with classification

- Cluster the training dataset into non-overlapping subsets $\mathcal{D}_1$ and $\mathcal{D}_2$, where the behavior of function is smoother
- Construct global PC expansions $g_i(\boldsymbol{x}) = \sum_k c_{ik}\Psi_k(\boldsymbol{x})$ using each dataset individually ($i = 1, 2$)
- Declare a surrogate

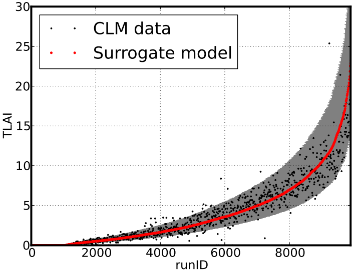
$$g_s(\boldsymbol{x}) = \begin{cases} g_1(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in^* \mathcal{D}_1 \\ g_2(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in^* \mathcal{D}_2 \end{cases}$$

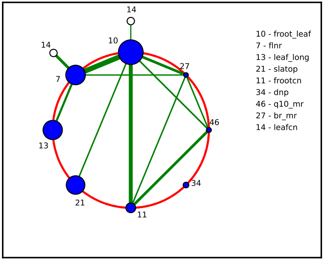
$^*$ Requires a classification step to find out which cluster $\boldsymbol{x}$ belongs to. We applied Random Decision Forests (RDF).

- Caveat: the sensitivity information is harder to obtain.

# Sparse PC surrogate for the Community Land Model

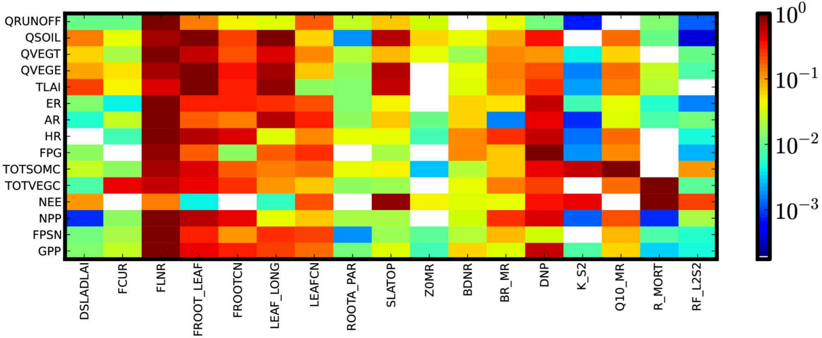
- Main effect sensitivities : rank input parameters
- Joint sensitivities : most influential input couplings
- About $200$ polynomial basis terms in the $50$-dimensional space
- Sparse PC will further be used for
  - sampling in a reduced space
  - parameter calibration against experimental data



# Sparse PC surrogate for the Community Land Model

- Main effect sensitivities : rank input parameters
- Joint sensitivities : most influential input couplings
- About $200$ polynomial basis terms in the $50$-dimensional space
- Sparse PC will further be used for
  - sampling in a reduced space
  - parameter calibration against experimental data



# Summary

- Surrogate models are necessary for complex models
  - Replace the full model for both forward and inverse UQ
- Uncertain inputs
  - Polynomial Chaos surrogates well-suited
- Limited training dataset
  - Bayesian methods handle limited information well
- Curse of dimensionality
  - The hope is that not too many dimensions matter
  - Compressive sensing (CS) ideas ported from machine learning
  - We implemented *iteratively* reweighting Bayesian CS algorithm that reduces dimensionality and increases order on-the-fly.

---

- Open issues
  - Computational design. What is the best sampling strategy?
  - Overfitting still present. Cross-validation techniques help.

# Literature


- M. Tipping, “Sparse Bayesian learning and the relevance vector machine”, *J Machine Learning Research*, 1, pp. 211-244, 2001.
- S. Ji, Y. Xue and L. Carin, “Bayesian compressive sensing”, *IEEE Trans. Signal Proc.*, 56:6, 2008.
- S. Babacan, R. Molina and A. Katsaggelos, “Bayesian compressive sensing using Laplace priors”, *IEEE Trans. Image Proc.*, 19:1, 2010.
- E. J. Candes, M. Wakin and S. Boyd. “Enhancing sparsity by reweighted $\ell_1$ minimization”, *J. Fourier Anal. Appl.*, 14 877-905, 2007.
- A. Saltelli, “Making best use of model evaluations to compute sensitivity indices”,*Comp Phys Comm*, 145, 2002.

- K. Sargsyan, C. Safta, H. Najm, B. Debusschere, D. Ricciuto and P. Thornton, “Dimensionality reduction for complex models via Bayesian compressive sensing”, *Int J for Uncertainty Quantification*, 4(1), pp. 63-93,2014.

- J. Jakeman, M. Eldred and K. Sargsyan, “Enhancing $\ell_1$-minimization estimates of polynomial chaos expansions using basis selection”, *J Comp Phys*, in press, 2015, see ArXiv.

# Random variables represented by Polynomial Chaos

$$X \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta})$$

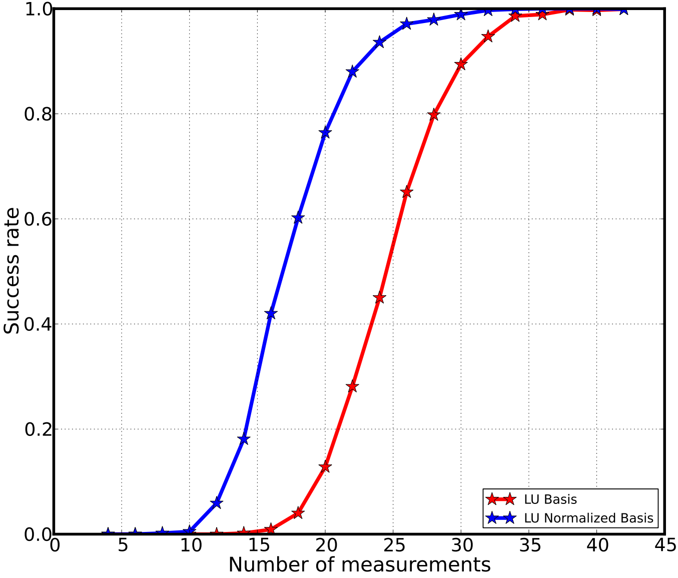
- $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_d)$ standard i.i.d. r.v.
  $\Psi_k$ standard polynomials, orthogonal w.r.t. $\pi(\boldsymbol{\eta})$.

$$\Psi_k(\eta_1, \eta_2, \ldots, \eta_d) = \psi_{k_1}(\eta_1)\psi_{k_2}(\eta_2)\cdots\psi_{k_d}(\eta_d)$$

- Typical truncation rule: total-order $p$, $k_1 + k_2 + \ldots k_d \leq p$.
  Number of terms is $K = \frac{(d+p)!}{d!p!}$.
- Essentially, a parameterization of a r.v. by deterministic spectral modes $c_k$ .
- Most common standard Polynomial-Variable pairs:
  (continuous) Gauss-Hermite, Legendre-Uniform,
  (discrete) Poisson-Charlier.

[Wiener, 1938; Ghanem & Spanos, 1991; Xiu & Karniadakis, 2002; Le Maître & Knio, 2010]

# Basis normalization helps the success rate



# Strong discontinuities/nonlinearities challenge global polynomial expansions

- Basis enrichment [Ghosh & Ghanem, 2005]
- Stochastic domain decomposition
  - Wiener-Haar expansions,
    Multiblock expansions,
    Multiwavelets, [Le Maître *et al*, 2004,2007]
  - also known as Multielement PC [Wan & Karniadakis, 2009]
- Smart splitting, discontinuity detection
  [Archibald *et al*, 2009; Chantrasmi, 2011; Sargsyan *et al*, 2011; Jakeman *et al*, 2012]
- Data domain decomposition,
  - Mixture PC expansions [Sargsyan *et al*, 2010]
- Data clustering, classification,
  - Piecewise PC expansions

# Sensitivity information comes free with PC surrogate,

$$g(x_1, \ldots, x_d) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i)]}{Var[g(\boldsymbol{x})]} = \frac{\sum_{k \in \mathbb{I}_i} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

$\mathbb{I}_i$ is the set of bases with only $x_i$ involved

# Sensitivity information comes free with PC surrogate,

$$g(x_1, \ldots, x_d) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i)]}{Var[g(\boldsymbol{x})]} = \frac{\sum_{k \in \mathbb{I}_i} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

- Joint sensitivity indices

$$S_{ij} = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i, x_j)]}{Var[g(\boldsymbol{x})]} - S_i - S_j = \frac{\sum_{k \in \mathbb{I}_{ij}} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

$\mathbb{I}_{ij}$ is the set of bases with only $x_i$ and $x_j$ involved

# Sensitivity information comes free with PC surrogate,

## but not with piecewise PC

$$g(x_1,\dots,x_d) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i)]}{Var[g(\boldsymbol{x})]} = \frac{\sum_{k\in\mathbb{I}_i} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

- Joint sensitivity indices

$$S_{ij} = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i,x_j)]}{Var[g(\boldsymbol{x})]} - S_i - S_j = \frac{\sum_{k\in\mathbb{I}_{ij}} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

- For piecewise PC, need to resort to Monte-Carlo estimation [Saltelli, 2002].